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An application of a theorem of Rothmaler

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Abstract

In this article, the notion of purely large structure is introduced. It is shown, with the aid of a Theorem of Rothmaler, that any finitely accessible class possesses purely large structures. This applies to the class $Mod(R)$ of all left modules over a given ring R. The theory T^* of purely large modules is always complete. It is shown that T^* is model-complete if and only if R is regular. For any algebra of finite representation type R, over an infinite field, T^* is axiomatizable by one sentence over $Th(Mod(R))$. A characterization of pure semisimple rings, in terms of purely large modules, is obtained.

Keywords: Purely large structure, finitely accesible class, largest complete theory.

1 Purely large structures

We recall some basic notions and facts from model theory. For a complete discussion of these topics see [2, 3, 5]. For model theory of algebraic systems we refer to [6]. Let L be a first-order language. Two $L-$structures A and B are called elementarily equivalent (notation : $A \equiv B$) if A and B satisfy the same first order sentences in L (i.e. $Th(A) = Th(B)$). A class Σ of $L-$structures is called elementarily closed if $A \equiv B, B \in \Sigma$, implies $A \in \Sigma$. A class Σ is called axiomatizable if Σ can be defined by a family of first order sentences in L. Further, Σ is called finitely axiomatizable if it can be defined by a first order sentence in L. We let $Th(\Sigma)$ be the set of all sentences true in each structure in Σ. A homomorphism $f : A \to B$ is said to be pure if for any positive primitive(or pp for short) formula $\phi(\overline{x})$ and any tuple $\overline{a}$ from A, the validity of $\phi(f(\overline{a}))$ in B entails that of $\phi(\overline{a})$ in A. Note that every pure map is an isomorphic embedding, therefore, these maps are also called pure embeddings. A substructure A of a structure B is called pure if the inclusion of A in B is pure. Let A be an $L-$ structure and K be a class of $L-$structures. Then A is said to be absolutely pure in K if every embedding of A into a structure from K is pure. Suppose T_1, T_2 are complete theories in L. The following conditions are equivalent:

(i) Every model of T_1 is purely embedded in some model of T_2.
(ii) Some model of T_1 is purely embedded in some model of T_2.

To show that (ii) $\Rightarrow$ (i), suppose M is a model of T_1 and M purely embeds in some model N of T_2. If X is any model of T_1, then $X \equiv M$. By Frayne's Lemma [2, Ch. 8, Lemma 1.1], there is an ultrafilter pair (I, u) such that X is elementarily embeddable in the ultrapower M^I/u. Note that M^I/u purely embeds in $N^I/u \equiv N$ and so X is purely embedded in some model of T_2. Write $T_1 \leq T_2$ if the preceding equivalent conditions are satisfied.

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Let K be a class of L-structures and $S(K)=\{Th(X):X\in K\}$. Note that $(S(K),\leq)$ is a preordered set. One may define an equivalence relation $\sim$ on $S(K)$ such that $T_1\sim T_2$ iff $T_1\leq T_2$ and $T_2\leq T_1$. Using this relation it is possible to construct a partial order on the set of equivalence classes $S(K)/\sim$. We can define $[T_1]\blacktriangleleft[T_2]$ iff $T_1\leq T_2$.

In this article, a structure $M\in K$ is said to be purely large in K if given any $X\in K$, there exists a pure embedding $f:X\rightarrow Y$, $Y\equiv M$. This means that every member of K is purely embedded in some model of $T^*=Th(M)$. Hence, for all $X\in K$, $Th(X)\leq T^*$ in $S(K)$.

In the case of modules, existence of purely large modules in the class $Mod(R)$ of all left R-modules, follows from a Theorem of Ziegler [14], which states that each module is elementarily equivalent to a direct sum of algebraically compact indecomposable modules. In fact, if I_o is the set of algebraically compact indecomposable modules in $Mod(R)$ and $D=\oplus\{M:M\in I_o\}$, then $D^{(\aleph_o)}$ is a purely large module in $Mod(R)$:

Let $X\in Mod(R)$. Then $X\equiv\bigoplus_{j\in J}M_j$, for some set J, $M_j\in I_o$, for all $j\in J$.

By Frayne's Lemma [2, Ch. 8, Lemma 1.1], there is an ultrafilter pair (I,u) such that X is elementarily embeddable in the ultrapower $(\bigoplus_{j\in J}M_j)I/u$.

Note that each M_j is a direct summand of D, so $\bigoplus_{j\in J}M_j$ is isomorphic to a direct summand of $D^{(\alpha)}$, where $\alpha=\sup(\aleph_o,card(J))$. It follows that X is purely embeddable in the ultrapower of $D^{(\alpha)}$ modulo u. Observe that the last module is elementarily equivalent to $D^{(\aleph_o)}$.

We note also that if $M=\oplus\{M_T:T$ is a complete theory of modules, M_T is a chosen model of $T\}$ and $T^*=Th(M)$, then $T\leq T^*$, for any complete theory of modules T [7, Proposition 2.32]. In this case T^* is the largest complete theory of modules and M is a purely large module in $Mod(R)$.

Let K be a finitely accessible class of L-structures in the sense of Adamek and Rosicky [1], i.e. it is closed under direct limits and contains a set P of finitely presented structures such that each member from K is a direct limit of structures from P (see also [5, Ch.9]. The main objective of this article is to show that any finitely accessible class K of L-structures possesses a purely large structure C. This gives rise to the largest complete theory $T^*=Th(C)$. But first some more notation. Given a poset $I=(I,\leq)$ and $i\in I$, let $I_{\geq i}$ denote the set $\{j\in J:j\geq i\}$. Clearly, $(I,\leq)$ is directed if and only if the set F_o of all subsets of I which contain a set $I_{\geq i}$ for some $i\in I$ form the Frechet filter on I. Given a direct system $D=(A_i,h_{ij})$, $i,j\in I, i\leq j$, over a directed set $I=(I,\leq)$ and a filter F on I containing the Frechet filter F_o, let lim D denote the direct limit of the A_i, A their direct product $\prod_I A_i$ and A/F their reduced product modulo F. We recall the following Theorem of Ph. Rothmaler [9, Corollary 7.4] or [8, Proposition 3.1].

Theorem 1
(In the above notation) whenever the filter F contains the Frechet filter F_o on I, there is a canonical embedding of lim D in the reduced product A/F, and this embedding is a pure embedding.

As an application of the preceding theorem, one can obtain the following result.

Corollary 2
Let K be a finitely accessible class of L-structures , P_o be the set of finitely presented structures in K and $C=\Pi\{M:M\in P_o\}$. Suppose that each object in P_o contains the one element structure 1_L as a substructure. Then C is a purely large structure in K.

Proof. Each member X from K is a direct limit of structures from P_o, say $X=\lim D$, where $D=(N_i,h_{ij}), i,j\in I, i\leq j$. Let u be a non-principal ultrafilter on I such that u contains the Frechet filter F_o. By Theorem 1, there is a pure embedding of X in the ultraproduct $\Pi N_i/u$. Since each object in P_o contains the one element structure 1_L as a substructure, each N_i purely embeds in C [9, Remark 6.5]. So there exists a pure embedding from $\Pi N_i/u$ into the ultrapower C^I/u of C. It follows that X purely embeds in $C^I/u \equiv C$.

Remark 3
Let A and B be L-structures. We say that A is *pp*-elementarily equivalent to B if each *pp* sentence of L true in A is also true in B. We say that a theory is *pp*-complete if all its models are *pp*-elementarily equivalent.

Let K be a finitely accessible class of L-structures with a purely large structure C. Then $Th(K)$ is *pp*-complete:

For any A , B in K, there are $\overline{A}$, $\overline{B}$ in K and pure embeddings f and g such that

$$\begin{array}{ccc} \overline{A} & \equiv C \equiv & \overline{B} \\ f\uparrow & & \uparrow g \\ A & & B \end{array}$$

Since *pp* sentences are preserved by homomorphisms, it is clear that for any *pp* sentence σ of L, σ holds in A if and
only if σ holds in B. Hence K has exactly one model (up to *pp*-elementary equivalence).

2 Purely large modules

In this section, R denotes a ring, 'modules' mean left R-modules and $Mod(R)$ is the class of all modules over R. We note that Corollary 2 can be applied to the class $Mod(R)$. As a general reference on model theory of modules we use [7].

Let Δ be the class of all purely large modules in $Mod(R)$. The following facts (in the next Remark) are essential.

Remark 4
(1) Δ is an axiomatizable class.
(2) $T^* = Th(\Delta)$ is complete.
(3) Δ is closed under direct products.
(4) Δ is closed under reduced products.
(5) Δ is preserved by pure extensions.
(6) (a) Δ is preserved by pure submodules iff (b) Every non-zero module is purely large iff (c) The theory of non-zero R-modules is complete.

Proof. (1) Let $M_i \in \Delta$ for all $i \in I$ and u any non-principal ultrafilter on I. Any module X purely embeds in $Y_i \equiv M_i$. Since $X \prec X^u$ and the last module purely embeds in $\Pi Y_{i/}/u \equiv \Pi M_i/u$, then $\Pi M_i/u \in \Delta$ and so Δ is closed under ultraproducts. One can easily see that is Δ elementarily closed. Hence Δ is an axiomatizable class [3, Corollary 6.1.16].

(2) It follows from Theorem 4 of [11].

(3) Since X purely embeds in the direct power X^I, then, as in (1), we can show that Δ is closed under direct products.

(4) By Theorem 2.10 of [12], any reduced product $\Pi M_i/F$, F is a filter on I, is elementarily equivalent to a direct product of ultraproducts of the modules M_i, $i \in I$. Now (4) follows from (1) and (3).

(5) Obvious.

(6) (a) $\Longrightarrow$ (b) and (c) $\Longrightarrow$ (a) follow from (1). (b) $\Longrightarrow$ (c) follows from (2).

In [10], the notion of large module 'gros module' was introduced : an R-module M is said to be large in $Mod(R)$ if given any module X, there exists an isomorphic embedding $f: X \to Y$, $Y \equiv M$. Note that M is large iff every consistent existential sentence (in the first order language of modules over R) holds in M[10]. It follows that the class Γ of all large modules in $Mod(R)$ is an axiomatizable class. It was shown in [10] that $M_o = \bigoplus_{j \in J} E_j^{(\aleph_o)}$, where $\{E_j : j \in J\}$ is the set of all injective envelopes of cyclic modules over R, is a large module.

The concept of $\aleph_o$-injective module, due to Eklof and Sabbagh [4], played an important role in model theory of modules. An R-module M is said to be $\aleph_o$-injective if for every finitely generated ideal I any homomorphism of I into M can be extended to a homomorphism of R into M. We note that the large module $M_o = \bigoplus_{j \in J} E_j^{(\aleph_o)}$ is $\aleph_o$-injective [4, Proposition 3.10]. If R is left coherent, then $Th(M_o)$ is a model-companion of the theory T_R of R-modules [4, Theorem 4.8]. A characterization of regular rings, in terms of $\aleph_o$-injective modules, was given. Namely, a ring R is regular in the sense of Von Neumann iff every R-module is $\aleph_o$-injective iff every R-module is absolutely pure [4, Proposition 3.25]. It follows that if R is regular then $\Delta = \Gamma$. We prove the converse for left coherent rings.

Theorem 5
For a left coherent ring R, the following assertions are equivalent:

(i) R is regular.
(ii) $\Delta = \Gamma$.
(iii) $Th(\Gamma)$ is complete.

Proof. It remains to show that (iii) $\Rightarrow$ (i): Let $C \in \Delta \subseteq \Gamma$. Since $M_o \in \Gamma$ and $Th(\Gamma)$ is complete, $C \equiv M_o$. We note that the $\aleph_o$-injective modules over a left coherent ring constitute an axiomatizable class [4, Theorem 3.16]. It follows that C is $\aleph_o$-injective. Thus, every R-module X purely embeds in an $\aleph_o$-injective module Y. Now the sequence $O \to X \to Y \to Y/X \to O$ is pure exact and so $X \oplus Y/X \equiv Y$ [7, Lemma 2.23]. Hence, $X \oplus Y/X$ is $\aleph_o$-injective, and so is X [4, Prop.3.9]. Therefore R is regular.

Generally, T^* is not model-complete. However, we have the following result

Theorem 6
A ring R is regular if and only if T^* is model-complete.

Proof. The 'only if part' follows from Theorem 2 of [10]. Conversely, suppose T^* is model-complete.

Since $T_R \subseteq T^*$ and every model of T_R embeds in a model of T^*, then T^* is a model-companion of T_R. Thus, R is left coherent [4, Theorem 4.1]. Obseve that $Th(M_o)$ is also a model-companion of T_R. Hence, T^* and $Th(M_o)$ are logically equivalent, and so $M_o \in \Delta$. It follows that every module in Δ is $\aleph_o$-injective and consequently every module is $\aleph_o$-injective. Therefore R is regular.

In the sequel, purely large modules are used to characterize pure semisimple rings.

Theorem 7
For any ring R, the following statements are equivalent:

(i) Every module in Δ is pure-injective.
(ii) Some, equivalently every, module in Δ is $\sum$-pure-injective.
(iii) R is left pure semisimple (i.e. every module is pure-injective).
(iv) Every module in Δ is pure-projective.

PROOF. (i) $\Rightarrow$ (ii) : First we note that if I is an infinite set, then $X^I \equiv X^{(I)} \equiv X^I/X^{(I)}$ for any R-module X [6, Proposition 6.17]. For any $X \in \Delta$, $X^{(\aleph_o)} \equiv X^{\aleph_o} \in \Delta$. Hence, $X^{(\aleph_o)}$ is pure-injective, and so X is $\sum$-pure-injective [6, Thm.7.12(i)]. (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are evident. (iv) $\Rightarrow$ (i): Let $X \in \Delta$. Since $X^I/X^{(I)} \equiv X^I \in \Delta$, then under the hypothesis (iv), $X^I/X^{(I)}$ is pure-projective. It follows that the following pure-exact sequence splits,

$$O \to X^{(I)} \to X^I \to X^I/X^{(I)} \to O.$$

This means that the natural embedding $X^{(I)} \subseteq X^I$ splits and so X is $\sum$- pure-injective [6, Theorem 7.12(iii)].

THEOREM 8

Let R be a finite dimensional algebra over an infinite field K. If R is of finite representation type, then Δ is a finitely axiomatizable class; i.e. $T^* = Th(\Delta)$ is axiomatizable by one sentence over $Th(Mod(R))$.

PROOF. Suppose R is of finite representation type and $\{M_1, \ldots, M_t\}$ is a family of representatives of non-isomorphic indecomposable objects in $Mod(R)$. It follows from [11, Theorem 1] that $Mod(R)$ has finitely many non-elementarily equivalent objects $\{H_1, \ldots, H_n\}$.

Let $M = M_1 \oplus \ldots \oplus M_t$. Of course, $M \in \Delta$ and $M \equiv H_k$ for some k, say $M \equiv H_1$. Since Δ is an axiomatizable class, it remains to show that its complement is closed under ultraproducts [3, Corollary 6.1.16]. Let $(X_i)_{i \in I}$ be any family of R-modules and let u be any non-principal ultrafilter on I. Suppose $X_i \notin \Delta$, for all $i \in I$. Let $I = I_2 \cup \ldots \cup I_k$, where $I_k = \{i \in I : X_i \equiv H_k\}$, $2 \le k \le n$.

Since u is an ultrafilter, there is a unique m, $2 \le m \le n$ such that $I_m \in u$. It follows that $\Pi X_i/u \cong H_m^{I_m}/u_m \equiv H_m$, where $u_m = \{J \cap I_m : J \in u\}$ is an ultrafilter over I_m. Since H_m is not elementarily equivalent to M, then $\Pi X_i/u \notin \Delta$ and so Δ is a finitely axiomatizable class.

REMARK 9

There is an easy proof of the preceding Theorem using the Ziegler spectrum. Let $M_1, \ldots, M_n$ be the indecomposable R-modules. Since R is of finite type the spectrum is discrete: choose, for each i, an isolating open neighbourhood (ϕ_i/ψ_i) for M_i. Then it is easy to see that the sentence $|\phi_1/\psi_1| \neq 1 \wedge \ldots \wedge |\phi_n/\psi_n| \neq 1$ axiomatizes Δ.

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